

## ON VERSION OF FARKAS LEMMA OF ALTERNATIVE

A.I. Golikov

*Dorodnicyn Computing Centre of RAS*

The paper provides a version of Farkas lemma of alternative linear systems, when the alternative systems having different matrices of various number of dimensions.

**Key words:** Farkas lemma, alternative linear systems, unconstrained optimization

Let the system determining a set  $X$  take the form of

$$Ax = b, \quad x \geq 0_n. \quad (I)$$

where  $A$  is matrix  $m \times n$ , vector  $b \in R^m$ ,  $\|b\| \neq 0$ .

The alternative system determining set  $U$  can be presented as

$$A^\top u \leq 0_n, \quad b^\top u = \rho > 0, \quad (II)$$

where  $\rho$  — an arbitrary fixed positive constant.

One, and only one of these systems, either (I) or (II), is always consistent, but never both. In case expression  $b^\top u > 0$  is used for (II) the above statement is known as Farkas lemma [1].

Besides their theoretical importance the theorems of alternative are of considerable value for computations [2].

For a given linear system, an alternative system is constructed in the space whose dimension is equal to the number of equations and inequalities in the original system (not counting constraints on the signs of variables). The original solvable system is solved by minimizing the residuals of the inconsistent alternative system. The results of this minimization are used to find the normal solution (with a minimal Euclidean norm) to the original system.

The replacement of the original problem by the minimization of the residuals of the inconsistent alternative system may be advantageous when the dimension of the new variables is less than that of the starting ones. In this case, such a reduction results in the minimization

problem in a space of lower dimension and allows one to obtain the normal solution to the original problem [2].

One and the same matrix  $A$  and vector  $b$  have always been used in alternative linear systems. The paper shows a different way of alternative systems involving application of different matrices with various dimensions, which can be advantageous from computational point of view.

To determine a system resolvability and to find a solvable problem solution it suffices to find just a single vector  $x^*$  or  $u^*$  of the below-mentioned problems of quadratic minimization over the positive orthant or unconstrained minimization of a piecewise quadratic function

$$\min_{x \in R_+^n} \frac{1}{2} \|b - Ax\|^2 = \frac{1}{2} \|b - Ax^*\|^2, \quad (1)$$

$$\begin{aligned} \min_{u \in R^m} \frac{1}{2} \{ \|(A^\top u)_+\|^2 + (\rho - b^\top u)^2 \} = \\ = \frac{1}{2} \{ \|(A^\top u^*)_+\|^2 + (\rho - b^\top u^*)^2 \}. \end{aligned} \quad (2)$$

The problems mutually dual to (1) and (2) respectively will be the below-listed strictly concave quadratic programming problems

$$\max_{z \in Z} \{ b^\top z - \frac{1}{2} \|z\|^2 \}, \quad Z = \{ z \in R^m : A^\top z \leq 0_n \}, \quad (3)$$

$$\begin{aligned} \max_{w \in W} \{ \rho w_2 - \frac{1}{2} \|w_1\|^2 - \frac{1}{2} w_2^2 \}, \\ W = \{ w_1 \in R_+^n, w_2 \in R^1 : Aw_1 - bw_2 = 0_m \}. \end{aligned} \quad (4)$$

From the dual features it follows that the solution  $z^*$  to problem (3) can be expressed through problem (1) solution by equation  $z^* = b - Ax^*$ ; taking this equation into account, one can obtain  $\|z^*\|^2 = b^\top z^*$  from the equality of the objective functions' optimal values.

Similarly the solution  $w_1^*, w_2^*$  to problem (4) can be expressed through the solution of problem (2) in the following way:  $w_1^* = (A^\top u^*)_+$ ,  $w_2^* = \rho - b^\top u^*$  and  $\|w_1^*\|^2 + w_2^{*2} = \rho w_2^*$  takes place.

If  $X \neq \emptyset$ , then  $w_2^* > 0$  and normal (with minimal Euclidean norm) solution to system (I) can be expressed through the solution of (2) as

follows:

$$\tilde{x}^* = (A^\top u^*)_+ / (\rho - b^\top u^*) = w_1^* / w_2^*. \quad (5)$$

If  $X = \emptyset$ , then  $\|z^*\| \neq 0_m$ , the normal solution to system (II) takes the form

$$\tilde{u}^* = \rho(b - Ax^*) / \|b - Ax^*\|^2. \quad (6)$$

The consideration below represents a special case of system (I) where matrix  $A$  has rank  $m$ , i.e.  $m \leq n$ . For the case concerned it will be shown that the system alternative to (I) can take a form different from (II), i.e. the alternative system can incorporate a matrix differing from  $A$  and a vector other than  $b$ .

If  $m \leq n$  then the system

$$Ax = b \quad (7)$$

is always solvable but its solutions may fail to include any nonnegative ones. Let  $\bar{X}$  denote the set of system (7) solutions. Note that set  $\bar{X}$  is always nonempty in contrast to set  $X$ . The general solution of the system of linear equations (7) can be written in the form

$$x = \bar{x} - K^\top y, \quad (8)$$

where  $\bar{x}$  is a particular solution of the system, and  $K^\top y$  is the general solution of the homogeneous system  $Ax = 0_m$ , and  $y \in R^\nu$ . The matrix  $K$  can be chosen to be any  $(\nu \times n)$  matrix such that its  $\nu$  rows form a basis of the null space of  $A$  where  $\nu = n - m$  is the defect of matrix  $A$ . Therefore,  $AK^\top = 0_{m\nu}$ . Here  $0_{ij}$  denotes  $(i \times j)$  matrix with zero entries.

Matrix  $K$  is not uniquely defined; it can be constructed in various ways. If we partition the matrix  $A$  as  $A = [B \mid N]$ , where  $B$  is non-degenerate, then we can represent  $K$  as  $K = [-N^\top (B^{-1})^\top \mid I_\nu]$ . If we reduce  $A$  by means of Gauss–Jordan transformations to the form  $A = [I_m \mid N]$ , then we can represent  $K$  as  $K = [-N^\top \mid I_\nu]$  [3].

Let us determine the set  $Y$  as

$$Y = \{y \in R^\nu : \bar{x} - K^\top y \geq 0_n\}. \quad (9)$$

Equation (8) can be considered as an affine mapping from  $R^\nu$  to  $R^n$ . Here the image of set  $Y$  is set  $X$  specified by system (I). There exists a one-to-one correspondence between  $X$  and  $Y$ .

Indeed, for any  $y \in Y$  equation (8) uniquely determines  $x \in X$ , i.e.

$$X = \bar{x} - K^\top Y \quad (10)$$

In case of a full-range overdetermined system (8) containing  $n$  linear equations and  $\nu$  variables  $y$  a pseudosolution

$$y = (KK^\top)^{-1}K(\bar{x} - x) = (K^\top)^+(\bar{x} - x), \quad (11)$$

always exists. It solves (8) and is unique if and only if  $\bar{x} - x \in \text{im } K^\top$ . This inclusion holds if and only if  $x \in \bar{X}$ . Thus, for any  $x \in \bar{X}$ , formula (11) determines an affine transformation that is the inverse of (8). Therefore, one can write

$$Y = (K^\top)^+(\bar{x} - X). \quad (12)$$

So the following two systems

$$Ax = b, \quad x \geq 0_n, \quad (I)$$

$$K^\top y \leq \bar{x}, \quad (I_y)$$

are either simultaneously solvable and interconnected by expressions (10) и (12) or simultaneously unsolvable if there exist no nonnegative general solution  $x = \bar{x} - K^\top y$  to system (I).

By Gale theorem [1] the following system determining set  $V$  will be alternative to system  $(I_y)$ ,

$$Kv = 0_\nu, \quad -\bar{x}^\top v = \rho > 0, \quad v \geq 0_n. \quad (II_v)$$

System  $(I_y)$  being equivalent to system (I), system  $(II_v)$  is simultaneously alternative to system (I).

The general solution to homogeneous system  $Kv = 0_\nu$  can be expressed by matrix  $A$  as  $v = -A^\top u$ . By changing the variables  $v = -A^\top u$ , one can present system  $(II_v)$  as follows:

$$A^\top u \leq 0_n, \quad b^\top u = \rho > 0. \quad (II)$$

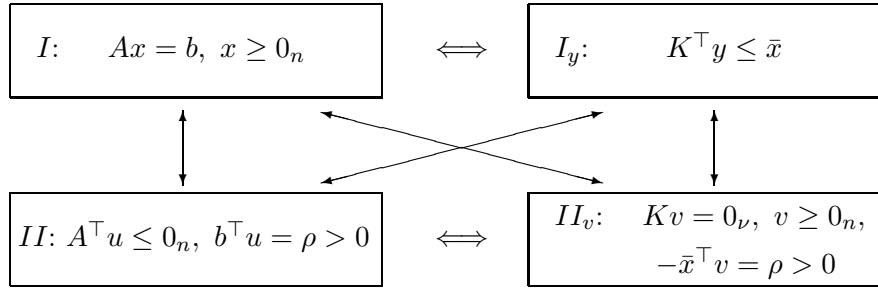
System  $(II)$  is alternative to  $(I)$ , hence to  $(I_y)$ .

If set  $V$  is nonempty then set  $U$  determined by system  $(II)$  is nonempty too, the two sets having a one-one mapping expressed by:

$$V = -A^\top U, \quad U = -(A^\top)^+ V,$$

where pseudoinverse matrix  $(A^\top)^+$  is as follows:  $(A^\top)^+ = (AA^\top)^{-1}A$ .

The alternative systems interrelation can be represented as follows:



The double arrows correspond to simultaneously solvable/unsolvable systems and the ordinary ones stand for alternative systems.

Let us provide linear programming interpretation of Farkas lemma. Here system  $(I)$  can be presented as the primal linear programming problem with its objective function coefficient vector identically equal to zero.

$$\min_{x \in R_+^n} \{0_n^\top x : Ax = b, \ x \geq 0_n\}. \quad (P)$$

The problem dual to  $(P)$  is as follows:

$$\max_{u \in R^m} \{b^\top u : A^\top u \leq 0_n\}. \quad (D)$$

It is common knowledge that for any couple of primal and dual LP problems there always exists one of the following four cases:

- 1) both primal and dual problems have solutions;
- 2) a primal problem is inconsistent and the dual one is unbounded;
- 3) a primal problem is unbounded and the dual one is inconsistent;

4) both primal and dual problems are inconsistent.

For problems  $(P)$  and  $(D)$  the latter two conditions cannot be fulfilled because the constraints in  $(D)$  are always consistent, vector  $u = 0_m$  is feasible.

The first two cases are only possible.

In case 1) the optimal values of goal functions for problems  $(P)$  and  $(D)$  are equal to zero and inequation  $b^\top u \leq 0$  holds for all feasible vectors  $u$  owing to the weak duality theorem. Hence it follows solvability of system  $(I)$  and unsolvability of system  $(II)$

$$A^\top u \leq 0_n, \quad b^\top u = \rho > 0. \quad (II)$$

In case 2) system  $(I)$  is inconsistent and system  $(II)$  is consistent for any  $\rho > 0$  due to unboundedness of dual problem  $(D)$ .

So one can obtain the simplest proof that  $(I)$  and  $(II)$  are alternative system employing a specific type of linear programming problems  $(P)$  и  $(D)$  and linear programming duality theory.

Problem (1) can be considered an auxiliary problem of penalty function method as applied to problem  $(P)$ . Problem (2) can be treated an auxiliary problem of Morrison method with its parameter being equal to  $\rho$  when applied to problem  $(D)$ .

Acknowledgment. The work was supported by the Russian Foundation for Basic Research, project 15-01-08259.

## Список литературы

- [1] Mangasarian O.L. *Nonlinear Programming*. - Philadelphia: SIAM, 1994.
- [2] Golikov A.I., Evtushenko Y.G. *Theorems of the Alternative and Their Applications in Numerical Methods* // Comput. Maths. and Math. Phys. 2003. V. 43 N° 3. Pp. 354–375.
- [3] Golikov A.I., Evtushenko Y.G. *Two Parametric Families of LP Problems and Their Applications* // Proceedings of the Steclov Institute of Mathematics, Suppl.1. 2002. Pp.S52-S66.